DUSO Mathematics League 2015 - 2016

SECTIONAL CHAMPIONSHIP

SOLUTIONS

S - 1. $\boxed{\mathbf{13}}$ Find u_1 first. Substituting, $u_1 = -3(3) + 7 = -2$, and now $u_2 = -3(-2) + 7 = \mathbf{13}$.

S - 2. $\boxed{\mathbf{(x-4)(x-1)(x^2+5x-4)}}$ This is equal to $x^4 - (5x-4)^2$, which factors as the difference of two squares: $(x^2 - (5x-4))(x^2 + (5x-4))$. The first factor can be rewritten but the second cannot, and the answer is $\mathbf{(x-4)(x-1)(x^2+5x-4)}$.

S - 3. $\boxed{\mathbf{125}\pi}$ The solid is a cylinder of radius 5 and height 5, so compute $\pi \cdot 5^2 \cdot 5 = \mathbf{125}\pi$.

S - 4. $\boxed{(\mathbf{1}, \mathbf{2})}$ The intersection point of the two lines will be on the line of reflection. Solve $2x + 2 = \frac{1}{2}x - 4$ to obtain $x = -4 \to y = -6$, so $(-4, -6)$ is on the line of reflection. Now, consider the x-intercept of the line $y = \frac{1}{2}x - 4$, which is $(8, 0)$. This point will be as far from $(-4, -6)$ as its image on $y = 2x + 2$ will be, because distance is preserved in a line reflection. Therefore, the distance from the image point $(x, 2x+2)$ to $(-4, -6)$ is $\sqrt{12^2 + 6^2} = \sqrt{180}$. Solve $\sqrt{(x+4)^2 + (2x+2+6)^2} = \sqrt{180}$ to obtain $x = -10$ or $x = 2$. If $x = -10$, then the line of reflection would have a negative slope, so we reject this and choose $x = 2 \to y = 6$. The midpoint of the segment connecting $(2, 6)$ and $(8, 0)$ is on the line of reflection, so $(5, 3)$ is on the line of reflection. The equation of the line passing through $(5, 3)$ and $(-4, -6)$ is $y = x - 2$. The ordered pair (a, b) is $(\mathbf{1}, \mathbf{2})$.

S - 5. $\boxed{\mathbf{-2}}$ The numerator can be factored by grouping to obtain $\frac{(x^2-4)(x-1)}{(x-2)(x-1)} = 0$. The values $x = 2$ and $x = 1$ yield indeterminate forms when substituted, so the only value of x which satisfies the equation is $x = \mathbf{-2}$.

S - 6. $\boxed{\mathbf{\frac{1}{105}}}$ There are 8 choices for the first digit, but then only 1 for the second. There are 6 choices for the third digit, but only 1 for the fourth. There are 4 choices for the fifth digit, but only 1 for the sixth. There are 2 choices for the seventh digit, but only 1 for the eighth. However, there is repetition among the digits, so the number of 8-digit numbers satisfying the conditions is $\frac{8 \cdot 6 \cdot 4 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2} = 4 \cdot 3 \cdot 2 \cdot 1$. This is a fraction of the $8!/(2 \cdot 2 \cdot 2 \cdot 2) = 2520$ different 8-digit numbers that can be made from those digits. The desired probability is $\frac{24}{2520} = \mathbf{\frac{1}{105}}$.

R-1. Compute the least positive integer value of x that satisfies $x^2 - 6.1x - 7.92 > 0$.
R-1Sol. $\boxed{\mathbf{8}}$ Recognize that the left hand side factors as $(x - 7.2)(x + 1.1) > 0$, so $x > 7.2$ implies that we choose **8**.

R-2. *Let N be the number you will receive.* A goat is tethered to the corner of a rectangular barn whose length is 10 meters and whose width is 4 meters. The tether is N meters long. Compute the goat's grazing area in square meters.
R-2Sol. $\boxed{\mathbf{52}\pi}$ The goat gets $\frac{3}{4}\pi N^2$ square meters off of one corner of the barn. If $N > 4$, then the goat gets another $\frac{1}{4}\pi(N-4)^2$ square meters. Substituting, we see that $N > 4$, so the grazing area is $\frac{3}{4}\pi 8^2 + \frac{1}{4}\pi 4^2$ or $\mathbf{52}\pi$ square meters.

R-3. *Let N be the number you will receive.* The circle centered at the origin with area N passes through two lattice points in the first quadrant: (A, B) and (B, A) where $A < B$. Pass back the ordered pair (A, B).
R-3Sol. $\boxed{(\mathbf{4}, \mathbf{6})}$ We have $A^2 + B^2 = N$, and substituting, we have $A^2 + B^2 = 52$. The only lattice points satisfying this are $(4, 6)$ and $(6, 4)$. Pass back $(\mathbf{4}, \mathbf{6})$.

R-4. *Let (A, B) be the coordinates you will receive.* The graph of the equation $y = A\cos x + B$ has a minimum at (C, D) where $0 < C \leq 2\pi$. Compute $C + D$.
R-4Sol. $\boxed{\pi + \mathbf{2}}$ The period is 2π, and the minimum will be at $x = 2\pi/2 = \pi = C$. Substituting, we have $D = 4\cos\pi + 6 = 2$, so our answer is $\pi + \mathbf{2}$.

R-5. *Let N be the number you will receive.* Circle O has diameter $\overline{AB}$. A circle P is inscribed in one of the semicircles formed by $\overline{AB}$. The semicircle has perimeter N. Compute the area of the inscribed circle.
R-5Sol. $\boxed{\frac{\pi}{4}}$ Let the area of the circle be r. The radius of the semicircle is $2r$, so the perimeter of the semicircle is $4r + 2\pi r$. Substituting, we have $r = 1/2$, and the area is $\pi(1/2)^2 = \frac{pi}{4}$.

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